CS 6505 : Computability and Algorithms Spring 2014

Randomized Approximation and Deterministic LP Rounding

Lecturer:Arindam Khan *Date: 14th March, 2014*

1 MaxCut

- $G = (V, E)$ is an undirected graph.
- A *cut* (S, T) is a partition of V.
- A *cut edge* is an edge with one end point in S and the other end point in T.
- The size of a cut is the number of cut edges.
- **MAX-CUT**: Find a largest size cut in G: NP-hard.
- Note: Finding a smallest size cut in G (called the *MIN-CUT* problem) can be solved in polynomial time using maximum flow algorithms.

2 Randomized Approximation Algorithm for MAX-CUT

2.1 Algorithm:

- For each $v \in V$, add v to a set H with probability 1/2.
- Let $T := V \backslash H$.
- The output cut is the set of edges between H and T.

2.2 Analysis:

- Let X_e be the random variable which is 1 if if the edge e is in the cut and 0 otherwise.
- What is $Pr[X_e = 1]$?
  - Let $e = uv$. Then e is in the cut if and only if either $u \in H$ and $v \in T$ or $u \in T$ and $v \in H$.
  - $Pr[X_e = 1] = Pr[(u \in H \cap v \in T) \cup (u \in T \cap v \in H)]$.
  - $Pr[X_e = 1] = Pr[(u \in H \cap v \in T)] + Pr[(u \in T \cap v \in H)]$.
  - $Pr[u \in H \cap v \in T] = Pr[u \in H] \cdot Pr[v \in T] = \frac{1}{2} \cdot \frac{1}{2} = 1/4$.
  - $Pr[u \in T \cap v \in H] = Pr[u \in T] \cdot Pr[v \in H] = \frac{1}{2} \cdot \frac{1}{2} = 1/4$.
  - Hence $Pr[X_e = 1] = 1/4 + 1/4 = 1/2$.

- **Expectation:** What is the expected value of X_e i.e, $\mathbb{E}[X_e]$?
  - $\mathbb{E}[X_e] = \sum x \times Pr[X_e = x] = 0 \times Pr[X_e = 0] + 1 \times Pr[X_e = 1] = Pr[X_e = 1] = 1/2.$
- Define the random variable $X := \sum_{e \in E} X_e$. What is $\mathbb{E}[X]$?

  ($\mathbb{E}[X]$ is the expected size of the cut produced by the algorithm.)
- **Linearity of Expectation**:
  Let $X_1.X_2, \ldots, X_n$ be discrete random variables and $X = \sum_{i=1}^{n} X_i$.
  Then $\mathbb{E}[X] = \mathbb{E}[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} \mathbb{E}[X_i]$.
- By Linearity of Expectation $\mathbb{E}[X] = \sum_{e \in E} \mathbb{E}[X_e] = |E|/2$.
- Deduce that in G there exists a cut of size at least $|E|/2$.
  - If all cuts have size $< |E|/2$, the average will also be $< |E|/2$.
  - We can take $|E|$ to be the upper bound on Opt.
  - Thus, $\mathsf{Opt}/2 \leq |E|/2 \leq ALGO \leq \mathsf{Opt} \leq |E|$
- The randomized algorithm is a 1/2 approximation algorithm in expectation. one can derandomize it using conditional expectations.
- For maximization problems, sometimes $\frac{1}{c}$-approximation algorithms are also called c-approximation algorithms for $c \geq 1$. So, the above algorithm is a 2-approximation algorithm in that notation.

3 Integer Linear Program for Set Cover

- Set of items: U.
- Collection of sets $\mathcal{S}$.
- Maximum frequency: f i.e., each item appears in at most f sets.
- Define variable x_S for each set $S \in \mathcal{S}$ such that $x_S = 1$ if $S \in \mathcal{S}$ and 0 otherwise.
- *Integer Linear Program (ILP):*
  - Objective: minimize $\sum_{S \in \mathcal{S}} c(S) x_S$
  - subject to: $\sum_{S: e \in S} x_S \geq 1, \quad (e \in U).$
  - $x_S \in \{0, 1\}, \quad (S \in \mathcal{S}).$
- The objective minimizes the cost of the set cover.
- The constraint make sure that for each item in U at least one set containing it be picked.
- *LP Relaxation:*
  - Objective: minimize $\sum_{S \in \mathcal{S}} c(S) x_S$
  - subject to: $\sum_{S: e \in S} x_S \geq 1, \quad (e \in U).$
  - $x_S \geq 0, \quad (S \in \mathcal{S}).$
- The LP can be solved in polynomial time, However the solution might be fractional.
- Note that the above LP is a relaxation of IPL i.e, any feasible solution of ILP is also a feasible solution of LP. But the converse might not be true. Thus, LP obejctive value $\leq$ the ILP objective value in case of minimization.
- We need to round the fractional values to an integral value to get a feasible solution for the integer program.

4 Deterministic Rounding: Algorithm

- Find an optimal solution to the LP relaxation.
- Return $\mathcal{C}$, the collection of all sets S with $x_S \geq 1/f$ in the solution.

5 Deterministic Rounding: Analysis

- Each element appears in at most f sets. So one of the sets must be picked to the extent of at least $1/f$ in the fractional cover. But then that set will be selected in the rounding.
- Thus $\mathcal{C}$ covers all the elements and is a valid set cover.
- The rounding increases x_S by a factor of at most f.
- Let LP be the value of the LP objective. Then, $LP \leq \mathsf{Opt}$ as Opt is the ILP objective value.
- $LP \leq \mathsf{Opt} \leq ALGO \leq f \cdot LP \leq f \cdot \mathsf{Opt}$.
- Thus the deterministic rounding gives f approximation.
- **Exercise:** Give an example where the algorithm gives $\Omega(f)$-approximation and thus the analysis is tight.
- Note: $O(\log n)$ and f are incomparable.

6 Other Techniques

- Randomized Rounding: $O(\log n)$-approximation for Set Cover.
- Primal Dual Schema: $O(\log n)$-approximation for Set Cover.
- Semidefinite Programming: 0.878-approximation for MaxCut.

7 Resources:

I am following chapter 1.2 (An introduction to the techniques and to linear programming: the set cover problem) and 1.3 (A deterministic rounding algorithm) from [1] for the lectures. The book is freely available online: *http://www.designofapproxalgs.com/* . You can also see chapter 14.1 *Rounding Applied to Set Cover* from [2]. Vertex cover is a special case of set cover with $f = 2$. In chapter 11.6 of the Kleinberg-Tardos textbook, similar LP rounding is given for 2-approximation for the vertex cover problem.

References

[1] Williamson, David P and Shmoys, David B. *The Design of Approximation Algorithms.* Cambridge University Press 2011.

[2] Vazirani, Vijay V. *Approximation Algorithms.* Springer 2001.